

Multi-Unit Facility Location Games

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Abstract. Motivated by applications in clustering and information retrieval, we extend the classical Hotelling setting to deal with multi-facility location duels. In the classical Hotelling setting, customers' locations are taken from the uniform distribution on the $[0, 1]$ segment, and there are two facility owners, each needs to decide on the location of her (single) facility, aiming to maximize the proportion of customers closer to it. We extend the Hotelling setting to *multi-unit facility location games*, where there are n players, where player i may control **several** facilities. We first analyze competition among the owner of k facilities to the owner of l facilities, for arbitrary (l, k) , where $l \leq k$. Our message for this extended Hotelling duel is quite striking: in **no** equilibrium of any such (l, k) facility location duel a facility will materialize in a location which is not part of the social welfare maximizing locations of the player who has k facilities, if she were to locate her facilities under no competition. This is obtained despite the lack of pure strategy equilibrium in any (l, k) duel whenever $l \neq k$. Moreover, for the n -player setting, we provide sufficient and necessary conditions for a pure strategy profile to be an equilibrium in such game. We then show that a pure-strategy equilibrium exist if and only if there is no dominant player who controls more than half of the facilities; in the latter case, under some conditions, a mixed strategy equilibrium of the form obtained in the (l, k) duel does exist.

1 Introduction

Consider two competing night clubs in a city, each of which wishes to attract audience to its nightly activities. The clubs' activities range from dancing to live shows, and each typical potential customer has her/his own favorite mixture between the two. While the actual services provided by the clubs are similar, they employ advertising campaigns emphasizing different mixtures of the above, with the aim of attracting customers. Indeed, typically customers will select the club whose promoted mix is closer to them. The game between the clubs is a classical facility location game, fundamental to the economics literature. However, if the clubs can segment/cluster the customers and have several campaigns (fixed number of campaigns for each player, determined by their budget), where each customer will select the club with the closest campaign to her/his interests, then the classical facility location setting should be widely generalized to allow for the study of the related competition, as will be discussed below.

Facility location has been a topic of major interest in the OR and CS literature for many years. It has been also of major interest to the data sciences, as the problem of clustering is closely related to the facility location problem, where centroids are interpreted as facility locations, and algorithms for finding them, such as k -means, are the standard practice [15]. On the other hand, in economics, facility location games have been introduced. The classical work by Hotelling [7] considers a duel between two parties who compete over consumers distributed uniformly over the interval $[0, 1]$; each party locates its facility on that interval, and grabs the proportion of the population closer to it. As it turns out, the only equilibrium in that setting is for both parties to locate their facility at $\frac{1}{2}$. Interestingly, while overwhelming many extensions of that basic setting exist (see related work below), the economic studies refer to competition between single-facility owners only, which make that work non-applicable to many applications, as the ones mentioned above (e.g. in clustering

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we are typically after selecting several centroids/clusters). Our aim is to expand the study of the Hotelling facility location duel to the general case where the players hold arbitrary k and l facilities, respectively.

To further motivate the above foundational study, consider strategic behaviors of publishers in an information retrieval [IR] context (see [2] for some formal definitions). Indeed, the publishers' competition game can be viewed as a problem of facility location in competitive settings, as originally introduced by Hotelling. If one views each customer as a reader whose interests are a trade-off between the amount of news versus the amount of articles in a newspaper, and sellers as publishers who control the amount of news versus articles to be published in their newspapers, we get an IR publishing game that coincides with the basic Hotelling setting. Needless to say, the IR setting is much broader, as customers and publishers' possibilities may be defined by quite arbitrary spaces, and various constraints on that setting may hold, but IR games are natural extensions of the above basic setting.

Assume a "strong" publisher who controls several outlets which it can maintain, e.g. two different Internet versions of its newspaper. This publisher can be viewed as being able to locate two "facilities" in the space of published data rather than only one; however, a "weak" publisher who can not maintain two such versions will need and be able to locate only one "facility". How would these different powers effect the behavior of the publishers? What would be optimal strategies for the different publishers? This is a novel challenge and question, which illustrates how valuable and deep the understanding of the above games may be for theory and practice.¹

Another motivation is professional forecasting (see [11]). In this setting, a set of forecasters wish to convince the market that they are well informed, as they try to predict a macroeconomic indicator $\xi \in \mathbb{R}$, which have uniform prior distribution in $[a, b]$. The forecasters have private information on the distribution of the state of ξ - they know that it lies somewhere in $[\xi_0, \xi_0 + 1]$ (where $[\xi_0, \xi_0 + 1] \subset [a, b]$). The multi-unit version of this game can be seen as a competition between investment houses (facility owners), employing several forecasters (facilities) each, with the goal to maximize the probability of having the "best" expert (the expert with the closest guess to the real value of ξ).

1.1 The Basic Model

Consider the following setting:

1. Player 1 has l facilities, and player 2 has k facilities. WLOG we assume $l \leq k$. The players need to decide where on the interval $[0, 1]$ they would locate their facilities. We refer to the facilities' location of player i as the strategy of i , denoted by x_i .
2. Customers are distributed uniformly² along the segment $[0, 1]$.
3. Each customer selects his nearest facility, where we break ties by unbiased coin flips. We refer to the proportion of customers selecting a player i 's facility as player i 's expected payoff / utility. Given the strategy profile (x_1, x_2) we denote the expected utility of player i by $U^i(x_1, x_2)$. Note that this is a fixed sum game.
4. Each player wants to maximize her expected payoff.

Throughout this paper, we address this setting as the " l vs k setting". This model can be generalized from several different angles: the number of players, the metric space they lay in, differentiating between the products (e.g by setting different prices), the payoff function, etc. In Sections 2,3 we consider the above foundational model; in Sect. 4 we deal with the extension to n players, addressed as "the multi-unit facility location game", in the sense that each player controls multiple facilities.

¹ See [9] for a study of other IR duels.

² Relaxation of this assumption will be discussed in Sect. 5.

1.2 Related Work

The model we adopt in service of initiating this study is the original model of Hotelling, phrased as a facility location duel, in the spirit of work on dueling algorithms [8]. A nice survey of the facility location games literature up to 1992 appears in [1]. For a more recent survey the reader is referred to [3]. While many extensions of the above basic setting have been introduced, in most of the existing literature the set of possible actions (i.e. possible facility locations) is very simple, e.g, a segment or a circle. For a recent discussion of general graph structures in that setting see [6]. None of that literature refers to players, each controlling several facilities to be located as their actions. In the algorithmic game theory literature the case we have only one entity that dictates the place of a facility (or several facilities), while the locations of customers are their private information and are strategically reported, emerged as a main source of interest [13,12]. In this context the players are the customers, while our work extends facility location games where the players are the facilities' owners.

1.3 Results

Our findings for the foundational l vs k problem are quite striking. We characterize the equilibria for customers arriving from the uniform distribution; we show that in no equilibrium a facility will be located in a place which is not selected in the optimal facility locations of the player who has the larger number of facilities, if she were alone and maximizing social welfare of the customers. Notice that this result is obtained although in all cases where the players control different number of facilities there is no pure strategy equilibrium.

More technically, we show the existence of equilibrium for the l vs k problem defined in Subsection. 1.1, in which player 1 does not contribute to the social welfare. In this equilibrium, it holds that if at time 0 player 2 (with k facilities) was the only server, and her goal was to maximize the social welfare, and at time 1 the other player would arrive - player 2 would stick with her original locations, and player 1 will locate her facilities in places that are already occupied with player 2's facilities. Then we show that these properties hold in any equilibrium. Namely, there is no equilibrium where more than k facilities are visible to the customers, and as a by product - the player that poses fewer facilities can be removed from the game without hurting the social welfare. We refer to such equilibria, where the largest player optimal monopoly facility locations are the only ones that materialize, as *monopoly-optimal* semi-mixed equilibria.

In Sect. 4 we study the extension to n players, where each player controls potentially more than one facility. We generalize the known necessary and sufficient conditions for a pure strategy profile to be in equilibrium (see [4]) for our setting. We then prove that the corresponding game, which we refer to as the *multi-unit facility location game*, possesses a pure strategy equilibrium if and only if it does not contain a dominant player, namely a player who controls more than half of the facilities (formally defined in Sect. 4.3). When such dominant player exist, we show that under some conditions a monopoly-optimal semi-mixed equilibrium exist, generalizing our finding for the l vs k duel. We also characterize the conditions for a strategy profile to be a pure strategy equilibrium.

Together, as multi-unit facility location games are initial game-theoretic extensions to the segmentation/clustering tasks, which captures competition among the parties who perform these tasks, our results characterize the structure of equilibria that can be obtained due to such competition.

2 l vs k Existence

Consider the game described above, where player 1 has l facilities and player 2 has k facilities ($l \leq k$).

First, for completeness, consider a scenario where only player 2 is in the game. It is quite straightforward to show the following:

Proposition 1. *The optimal facility locations for player 2 are³ :*

$$x_2 = \left(\frac{1}{2k}, \dots, \frac{2i-1}{2k}, \dots, \frac{2k-1}{2k} \right)$$

The proof is in the appendix. Our main aim is to see the ramifications of having the above duel.

2.1 l vs k - Equilibrium Protocol

Throughout the paper we use the term *protocol* to refer to a prescribed strategy profile provided to the players.

Protocol 1

Player 1: choose a tuple of size l uniformly from $\{\frac{1}{2k}, \frac{3}{2k}, \dots, \frac{2k-1}{2k}\}$. Denote this strategy $x_1^{p_{1k}}$.

Player 2: choose deterministic locations: $x_2^{p_{2k}} = (\frac{1}{2k}, \frac{3}{2k}, \dots, \frac{2k-1}{2k})$

Remark: if $l = k$, they both play the same strategy. Denote the strategies in this special case $x_1^{p_{kk}}, x_2^{p_{kk}}$.

In the rest of this section we will prove:

Theorem 1 (*Existence of equilibrium in the l vs k game*). *Protocol 1 is an equilibrium for the l vs k facility location game on $[0, 1]$ with uniform customers' distribution.*

To prove this theorem, we show that none of the players has greater payoff in case she deviates; this will be obtained using a series of lemmas. We first show:

Lemma 1. *In the l vs k game where $l \leq k$, in case player 2 plays according to Protocol 1, any deterministic strategy played by player 1 can lead to maximum expected payoff of $\frac{l}{2k}$.*

The proof is in the appendix. The following lemma deals with the k vs k game, for arbitrary k . In addition to being a basic building block for the general result, it also implies a unique deterministic equilibrium for the symmetric case.

Lemma 2. *In the k vs k game, if player 1 plays according to Protocol 1, any deterministic strategy $x_2 \neq x_2^{p_{kk}}$ played by player 2 will yield her less than $\frac{1}{2}$.*

Proof. First, notice that $U^2(x_1^{p_{kk}}, x_2^{p_{kk}}) = \frac{1}{2}$, since both players share the exact same locations. Next, Fix an arbitrary deterministic strategy x_2 such that $x_2 \neq x_2^{p_{kk}}$. With similar arguments to those given in Lemma 1, we conclude that each facility of player 2 may serve a proportion of not more than $\frac{1}{2k}$ of the customers, Therefore $U^2(x_1^{p_{kk}}, x_2) \leq \frac{1}{2}$. Assume by contradiction that $U^2(x_1^{p_{kk}}, x_2) = \frac{1}{2}$, and therefore each of player 2's facility contributes exactly $\frac{1}{2k}$. Denote $y_i = \frac{2i-1}{2k}$, and also

$$I_i = [y_i, y_{i+1}], \quad i \in \{1, 2, \dots, k-1\}$$

and note that under $x_1^{p_{kk}}$ player 1 has facilities in y_i , for $i \in \{1, 2, \dots, k-1, k\}$.

Step 1: Observe that no facility of player 2 is located in $I_0 = [0, \frac{1}{2k})$ or $I_k = (\frac{2k-1}{2k}, 1]$, since if it did - it would get strictly less than $\frac{1}{2k}$. Thus, all facilities of player 2 are placed in $\bigcup_{i=1}^{k-1} I_i$.

Step 2: If interval I_i contains exactly two facilities of player 2 (denoted b_l, b_r where $b_l < b_r$), they must lie on the endpoints of I_i , since:

³ By ease of notations, strategies are sometimes referred to as a set of facilities, and sometimes as a sorted tuple of ascending order.

1. If both of them are contained in the interior of I_i , then the contribution of b_l to player 2's payoff is:

$$\frac{b_r - y_i}{2} < \frac{y_{i+1} - y_i}{2} = \frac{1}{2k}$$

2. If b_l is located on the left endpoint while b_r is in the interior of I_i - then b_l contributes at most:

$$\underbrace{\frac{1}{2} \cdot \frac{1}{2k}}_{\text{customers from } I_{i-1}} + \underbrace{\frac{1}{2} \cdot \frac{b_r - b_l}{4}}_{\text{customers from } I_i} < \frac{1}{4k} + \frac{y_{i+1} - b_l}{4} = \frac{1}{2k}$$

And the same goes for the symmetric case.

Step 3: Interval I_i cannot contain 3 or more facilities of player 2, or otherwise -

1. The distance d between the leftmost facility and the rightmost facility inside I_i is less than $\frac{1}{k}$. In this case, a middle facility cannot get more than $\frac{d}{2} < \frac{1}{2k}$.
2. Two facilities of player 2 are located on the endpoints of I_i , along with at least one more facility in its interior. Therefore, the facilities on the endpoints get less than $\frac{1}{2k}$, using the same reasoning used in the previous step.

Step 4: If interval I_i for $i \leq i \leq k-1$ contains no facilities of player 2 at all, then there must be another interval I_{i^*} with two of her facilities. Since any facility placed in the interior of I_i will contribute $\frac{1}{2k}$, we can move one facility from I_{i^*} and put it in the middle of I_i without decreasing $U^2(x_1^{p_{kk}}, x_2)$, so w.l.o.g. we can assume all intervals contain at least one facility of player 2.

Step 5: According to steps 1-4, every interval contains either one facility of player 2 (type I) or two facilities of hers (type II). Since $x_2 \neq x_2^{p_{kk}}$, there must be at least one interval of type I. Since all k facilities of player 2 lie in I_1, \dots, I_{k-1} , the Pigeonhole principle implies that there should also be at least one interval of type II. It follows that there are two adjacent intervals of different types, which yields the desired contradiction. \square

Corollary 1. *In the k vs k game, the only equilibrium is when both players play according to Protocol 1.*

In order to analyze how well can player 2 respond to player 1's strategy for $l \leq k$, we introduce the following notation:

$\mu([a, b])$ – the expected number of player 1's facilities in the interval $[a, b]$

In general, μ is used to quantify the probability of seeing a facility of player 1 in some sub-segment. μ is a measure, and we shall use some of its properties.⁴ Note that if player 1 has l facilities then $\mu([0, 1]) = l$.

The following lemma uses the previous one; its proof exploits basic connections between k vs k duels and general l vs k duels.

Lemma 3. *In the l vs k game, if the strategy used by player 1 satisfies:*

$$\forall i \in \{1, 2, \dots, k\} : \mu(y_i) = \frac{l}{k} \quad \left(y_i = \frac{2i-1}{2k} \right)$$

then any deterministic strategy played by player 2 can lead to maximum payoff of $1 - \frac{l}{2k}$

⁴ See the appendix for a more formal treatment.

Remark: Note that $x_1^{p_{lk}}$ satisfies the above condition.

Proof. Denote X_1 as a random variable representing such strategy of player 1. It is easy to verify that $U^2(X_1, x_2^{p_{kk}}) = 1 - \frac{l}{2k}$. Next, fix an arbitrary deterministic strategy x_2 of player 2 such that $x_2 \neq x_2^{p_{kk}}$. Define -

$$\mathcal{V}_{y_i}(f_1, f_2)$$

to be the contribution of a facility located in y_i by player 1 to her payoff, when she selects deterministic locations f_1 and player 2 select deterministic f_2 (if $y_i \notin f_1$ it follows that $\mathcal{V}_{y_i}(f_1, f_2) = 0$). Note that for any set of locations A it holds that $\mathcal{V}_{y_i}(f_1, f_2) \geq \mathcal{V}_{y_i}(f_1 \cup A, f_2)$, since the proportion of customers that the facility located in y_i serves can only decrease. Thus:

$$U^1(X_1, x_2) = \sum_f U^1(f, x_2) \cdot \mathbb{P}(X_1 = f) = \sum_f \sum_{i=1}^k \left(\mathcal{V}_{y_i}(f, x_2) \cdot \mathbb{1}[y_i \in f] \cdot \mathbb{P}(X_1 = f) \right) \quad (1)$$

Since for any f such that $\mathbb{P}(X_1 = f) > 0$ it holds that $f \subset x_1^{p_{kk}}$, we derive from (1) that:

$$\begin{aligned} U^1(X_1, x_2) &\geq \sum_f \sum_{i=1}^k \left(\mathcal{V}_{y_i}(x_1^{p_{kk}}, x_2) \cdot \mathbb{1}[y_i \in f] \cdot \mathbb{P}(X_1 = f) \right) \\ &= \sum_{i=1}^k \mathcal{V}_{y_i}(x_1^{p_{kk}}, x_2) \underbrace{\left(\sum_f \mathbb{1}[y_i \in f] \cdot \mathbb{P}(X_1 = f) \right)}_{\mu(y_i)} \\ &= \frac{l}{k} \cdot \sum_{i=1}^k \mathcal{V}_{y_i}(x_1^{p_{kk}}, x_2) \\ &= \frac{l}{k} \cdot U^1(x_1^{p_{kk}}, x_2) \end{aligned} \quad (2)$$

It follows from Lemma 2 that $U^1(x_1^{p_{kk}}, x_2) > \frac{1}{2}$ for any $x_2 \neq x_2^{p_{kk}}$, hence

$$U^1(X_1, x_2) > \frac{l}{2k} \xrightarrow{1\text{-sum game}} U^2(X_1, x_2) < 1 - \frac{l}{2k} \quad (3)$$

So overall $x_2^{p_{kk}}$ is the only best response of player 2 against X_1 , achieving payoff of $1 - \frac{l}{2k}$, and the payoff of any other response of hers to X_1 would get strictly less than $1 - \frac{l}{2k}$. \square

Proof (of Theorem 1). Lemmas 1 and 3 imply that any change in player i 's strategy will not improve her expected payoff, yielding the desired result.

3 l VS k Quasi-Uniqueness

In this section we complete, when augmented with the results of the previous section, a main result of the paper. We show that any equilibrium of the above duels is monopoly-optimal. Namely, any equilibrium has the property that player 2 uses the previously mentioned equilibrium protocol, while player 1 uses some distribution over the set of locations selected by player 2. Moreover, based on the results of the previous section we characterize how this distribution may look like. While this does not imply uniqueness (as there are several strategies satisfying the conditions below), it

implies that player 1 does not contribute in any equilibrium any new location, beyond what player 2 contributes, which are the locations that correspond to the single-player k -facility social welfare maximizing assignment. We term this phenomenon quasi-uniqueness, as the location of the facilities is uniquely determined.

Lemma 4. *Properties of any l vs k equilibrium –*

If (x_1, x_2) is an equilibrium profile, then the following holds:

1. *Player 1 gets an expected payoff of $\frac{l}{2k}$*
2. $\mu([0, \frac{1}{2k})) = \mu((\frac{2k-1}{2k}, 1]) = 0$
3. $U^1(x_1, x_2) = U^1(x_1, x_2^{p_{lk}})$

The proof is in the appendix. Next, we present the main result in this section:

Theorem 2 (Quasi-Uniqueness of equilibrium in the l vs k game). *In the l vs k facility location duel on the $[0, 1]$ segment with uniform customers' distribution, the strategies (x_1, x_2) are in equilibrium iff:*

- C1:** x_1 satisfies that $\forall i \in \{1, 2, \dots, k\} : \mu(y_i) = \frac{l}{k}$
C2: $x_2 = x_2^{p_{lk}}$

Proof. We begin with the easier side - if **C1**, **C2** are satisfied, since player 2 has no benefit in deviating according to Lemma 3, and player 1 has the same inclination according to Lemma 1, we conclude that the profile (x_1, x_2) is in equilibrium.

On the other direction: Assume (x_1, x_2) is an equilibrium strategy profile for players 1 and 2. In case player 1 plays according to Protocol 1, the only best response for player 2 is to play according to the protocol as well. Thus if $x_1 = x_1^{p_{lk}}$ it follows that $x_2 = x_2^{p_{lk}}$, and since Protocol 1 satisfies **C1** and **C2** we are done.

If this not the case, then $x_1 \neq x_1^{p_{lk}}$. Next, we try to find strategy for player 2, x'_2 , such that $U^1(x_1, x_2) > U^1(x_1, x'_2)$, and using Lemma 4 it is equivalent to show that $U^1(x_1, x'_2) < U^1(x_1, x_2^{p_{lk}}) = \frac{l}{2k}$. By doing so, we show that player 2 will deviate in order to increase her payoff (and decrease player 1's). Using the fact that (x_1, x_2) is an equilibrium profile, and hence x'_2 should not exist - we can characterize the possible strategies x_1 .

step 1: $\mu(y_1) \geq \mu((y_1, y_2])$

Assume by contradiction that $\mu((y_1, y_2]) - \mu(y_1) = \delta > 0$. Consider

$$x'_2 = (y_1 + \epsilon_1, y_2 + \epsilon_0, y_3 + \epsilon_0, \dots, y_k + \epsilon_0)$$

for $\epsilon_1, \epsilon_0 > 0$ such that ⁵ :

$$\mu((y_1, y_1 + \epsilon_1)) < \frac{\delta}{6}, \quad \epsilon_0 < \frac{\delta \cdot \epsilon_1}{4}$$

Namely, x'_2 is similar to $x_2^{p_{lk}}$, but every facility is shifted right by ϵ_0 , except from the facility on y_1 which is shifted right by ϵ_1 . By Lemma 4.2 we are guaranteed that there no facilities of player 1 are in $[0, \frac{1}{2k}), (\frac{2k-1}{2k}, 1]$. Therefore, the payoff of player 1:

$$\begin{aligned} U^1(x_1, x'_2) &\leq U^1(x_1, x_2^{p_{lk}}) + \mu(y_1) \cdot \left(\frac{\epsilon_1}{2}\right) + \mu((y_1, y_1 + \epsilon_1)) \cdot (\epsilon_1) + \mu([y_1 + \epsilon_1, y_2 + \epsilon_0]) \cdot \left(\frac{\epsilon_0 - \epsilon_1}{2}\right) \\ &\leq \frac{l}{2k} - \frac{\epsilon_1}{2} \underbrace{\left(\mu([y_1 + \epsilon_1, y_2 + \epsilon_0]) - \mu(y_1) - 2 \cdot \mu((y_1, y_1 + \epsilon_1)) \right)}_{* \geq \frac{\delta}{2}} + \epsilon_0 \\ &\leq \frac{l}{2k} - \frac{\delta \cdot \epsilon_1}{4} + \epsilon_0 < \frac{l}{2k} \end{aligned}$$

⁵ We can assume that $\mu(y_i + \epsilon_j) = 0$ for $i \in \{1, \dots, k\}, j \in \{0, 1\}$ because of Lemma 8 in the appendix, and that such ϵ_0, ϵ_1 exist. We should also state that $\epsilon_0, \epsilon_1 < \frac{1}{2k}$.

And * is due to:

$$\begin{aligned}
& \mu([y_1 + \epsilon_1, y_2 + \epsilon_0]) - \mu(y_1) - 2 \cdot \mu((y_1, y_1 + \epsilon_1)) \\
& \geq \mu([y_1 + \epsilon_1, y_2]) - \mu(y_1) - 2 \cdot \mu((y_1, y_1 + \epsilon_1)) \\
& = \mu((y_1, y_2]) - \mu((y_1, y_1 + \epsilon_1)) - \mu(y_1) - 2 \cdot \mu((y_1, y_1 + \epsilon_1)) \\
& = \mu((y_1, y_2]) - \mu(y_1) - 3 \cdot \mu((y_1, y_1 + \epsilon_1)) \\
& > \delta - \frac{3 \cdot \delta}{6} = \frac{\delta}{2}
\end{aligned}$$

Hence we got a contradiction and $\mu(y_1) \geq \mu((y_1, y_2])$. By symmetry we also conclude that $\mu(y_k) \geq \mu([y_{k-1}, y_k))$.
step 2: $\forall 1 \leq i \leq k-2$: $\mu((y_i, y_{i+1}]) \geq \mu((y_{i+1}, y_{i+2}])$

Assume by contradiction that $\mu((y_{i+1}, y_{i+2}]) - \mu((y_i, y_{i+1}]) = \delta > 0$ holds for some i . Consider

$$x'_2 = (y_1 + \epsilon_0, \dots, y_i + \epsilon_0, y_{i+1} + \epsilon_1, y_{i+2} + \epsilon_0, \dots, y_k + \epsilon_0)$$

for $\epsilon_1, \epsilon_0 > 0$ such that:

$$\mu((y_{i+1}, y_{i+1} + \epsilon_1)) < \frac{\delta}{4}, \quad \epsilon_0 < \frac{\delta \cdot \epsilon_1}{4l}$$

Namely, x'_2 is similar to $x_2^{p_{lk}}$, but every facility is shifted right by ϵ_0 , except from the facility on y_{i+1} which is shifted right by ϵ_1 . The payoff of player 1:

$$\begin{aligned}
U^1(x_1, x'_2) & \leq U^1(x_1, x_2^{p_{lk}}) + \mu([y_1, y_1 + \epsilon_0]) \cdot \epsilon_0 + \mu((y_i + \epsilon_0, y_{i+1} + \epsilon_1)) \cdot \left(\frac{\epsilon_1 - \epsilon_0}{2}\right) \\
& \quad + \mu([y_{i+1} + \epsilon_1, y_{i+2} + \epsilon_0]) \cdot \left(\frac{\epsilon_0 - \epsilon_1}{2}\right) \\
& \leq \frac{l}{2k} + l \cdot \epsilon_0 - \frac{\epsilon_1}{2} \cdot \underbrace{\left(\mu([y_{i+1} + \epsilon_1, y_{i+2} + \epsilon_0]) - \mu((y_i + \epsilon_0, y_{i+1} + \epsilon_1))\right)}_{* \geq \frac{\delta}{2}} \\
& \leq \frac{l}{2k} + l \cdot \epsilon_0 - \frac{\delta \cdot \epsilon_1}{4} < \frac{l}{2k}
\end{aligned}$$

And * is due to:

$$\begin{aligned}
& \mu([y_{i+1} + \epsilon_1, y_{i+2} + \epsilon_0]) - \mu((y_i + \epsilon_0, y_{i+1} + \epsilon_1)) \geq \mu([y_{i+1} + \epsilon_1, y_{i+2}]) - \mu((y_i, y_{i+1} + \epsilon_1)) \\
& = \mu((y_{i+1}, y_{i+2}]) - \mu((y_{i+1}, y_{i+1} + \epsilon_1)) - \mu((y_i, y_{i+1}]) - \mu((y_{i+1}, y_{i+1} + \epsilon_1)) \\
& = \mu((y_{i+1}, y_{i+2}]) - \mu((y_i, y_{i+1}]) - 2\mu((y_{i+1}, y_{i+1} + \epsilon_1)) > \delta - 2 \cdot \frac{\delta}{4} = \frac{\delta}{2}
\end{aligned}$$

From symmetry we also conclude that

$$\forall 1 \leq i \leq k-2: \mu([y_i, y_{i+1})) \leq \mu([y_{i+1}, y_{i+2}))$$

step 3: From steps 1,2 and monotonicity of μ :

$$\begin{aligned}
\mu(y_1) & \geq \mu((y_1, y_2]) \geq \mu((y_2, y_3]) \dots \geq \mu((y_{k-2}, y_{k-1}]) \geq \mu((y_{k-1}, y_k]) \geq \mu(y_k) \\
\mu(y_1) & \leq \mu([y_1, y_2)) \leq \mu([y_2, y_3)) \dots \leq \mu([y_{k-2}, y_{k-1})) \leq \mu([y_{k-1}, y_k)) \leq \mu(y_k)
\end{aligned}$$

Therefore we can extract from these inequalities that:

1. $\forall i, j \in \{1 \dots k\}$: $\mu(y_i) = \mu(y_j) = \frac{\mu([0,1])}{k} = \frac{l}{k}$
2. $\forall i, j \in \{1 \dots k\}$: $\mu((y_i, y_{i+1})) = \mu((y_j, y_{j+1})) = 0$

So **C1** is satisfied. From Lemma 3 we conclude that in case player 2 plays strategy $x_2 \neq x_2^{p_{lk}}$ her payoff is less than $1 - \frac{l}{2k}$, hence **C2** is satisfied as well. \square

Corollary 2. *In l vs k facility location duel, the only possible equilibrium is monopoly optimal.*

4 General Multi-Unit Games

In this section, we analyze the n -player version of the game. We show that some of the multi-unit games do possess a pure Nash equilibrium, unlike the l vs k case (with $l \neq k$). First, we show the necessary and sufficient conditions for a profile to be in equilibrium, which provides a full characterization of the structure of equilibrium in case it exist. Later on, we show that not all multi-unit games possess a pure equilibrium, relying on the results achieved for the l vs k duel. We also show that a monopoly optimal semi-mixed strategy equilibrium can still be obtained for some of the games where no pure strategy equilibrium exist.

4.1 The Generalized Model

Consider the following model:

1. n players, where player i controls k_i facilities (w.l.o.g. we assume $k_1 \leq k_2 \leq \dots \leq k_n$), and the total number of facilities denoted K .
2. Customers are distributed uniformly along the segment $[0, 1]$.
3. Each customer selects his nearest facility, where ties are broken by unbiased coin flips.
4. Each player wants to maximize her expected payoff.

From now on we refer to this extension of the game as the *multi-unit facility location game* or simply the multi-unit game. We also introduce three concepts that will simplify the proofs in this section:

- A facility f located in l_f is called a *lone facility* in profile X if under X there is no other facility f' such that $l_f = l_{f'}$.
- A facility f located in l_f is called a *paired facility* in profile X if there is exactly one more facility f' such that $l_f = l_{f'}$.
- The proportion of customers traveling left in order to be served by a facility (or facilities) located in $a \in [0, 1]$ is denoted $c_r(a)$ (as they are right of a), and $c_l(a)$ is the proportion of customers served at a coming from its right.⁶

When the profile discussed is clear from the context we will omit it.

4.2 Pure Strategy Equilibrium Characterization

The following result summarizes what is known about pure equilibria in the original Hotelling game (the single-unit game):

Theorem 3. *For Hotelling game on the interval with uniform customer distribution (each player controls a single facility), necessary and sufficient conditions for a pure strategy profile to be in equilibrium are:*

1. *The two peripheral (leftmost/rightmost) facilities are paired.*
2. *No facility's payoff is smaller than $\max\{c_l(a), c_r(a)\}$ for any $a \in [0, 1]$.*

Proof. See [5].

The next definition is necessary in order to use the relations between the single-unit game and the multi-unit game:

⁶ If there is no facility in a then $c_l(a) = c_r(a) = 0$

Definition 1 (Flattened Game, Flattened Profile). Given a multi-unit game G with n players, where player j controls k_j facilities for $j \in \{1 \dots n\}$, and a pure strategy profile $X = (x_1, x_2, \dots, x_n)$:

1. The single-unit game with $K = \sum_{j=1}^n k_j$ players is called the flattened game and is marked G_f .
2. In the flattened game G_f , a pure strategy profile $Y = (y_1, y_2, \dots, y_K)$ is called the flattened profile of X if for every facility f located under X there exists i such that $y_i = f$. We denote this profile $\psi(X)$ (since G_f is symmetric game, several such profiles exist, but they are all identical up to the names of the players).

At this point, we want to show that the conditions of Theorem 3 are necessary conditions (not sufficient however) for pure equilibria in multi-unit games.

Lemma 5. In a multi-unit game G , a pure strategy profile $X = (x_1, x_2, \dots, x_n)$ is in equilibrium only if $\psi(X)$ is in equilibrium in the flattened game G_f .

Proof. Assume X is an equilibrium profile in G , and $\psi(X)$ is not in equilibrium in G_f . Thus, one of the conditions of Theorem 3 does not hold:

1. If the peripheral facilities are not paired, then there is a player in G who controls a lone peripheral facility. Moving this facility towards its neighbor would increase that player's payoff, hence she has a profitable deviation, in contradiction to X being in equilibrium in G .
2. In G_f , under $\psi(X)$, the payoff of facility j (of player i) is q , and there exist a location a with at least one facility such that $p = \max\{c_l(a), c_r(a)\}$ and $p - q = \delta > 0$. WLOG assume $p = c_l(a)$. There are three possible cases:
 - (a) None of the facilities located in a belong to player i . In this case, if she locates facility j in $a - \delta$ her payoff would increase by $p - \frac{\delta}{2} - q > 0$.
 - (b) Player i possess a lone facility in a - denote the location of its left neighbor b (it is lone hence not peripheral). Since $a - b = 2p$, player i can move facility j to $b + \delta$, and as a result her payoff would increase by $p - \frac{\delta}{2} - q > 0$.
 - (c) Two facilities are located in a , one of which belongs to player i : since $c_l(a) = c_r(a) = p$, (otherwise X is not an equilibrium), she can relocate facility j and the facility currently in a in $a - \frac{\delta}{2}$ and $a + \frac{\delta}{2}$. Her new payoff is greater since:

$$\left(p - \frac{\delta}{4}\right) + \left(p - \frac{\delta}{4}\right) - p - q = p - q - \frac{\delta}{2} > 0$$

hence X is not an equilibrium in G . □

Obviously, scenarios where a player control two neighboring facilities may result in a profitable deviation for her. The next lemma captures these situations:

Lemma 6. In a multi-unit game G , a pure strategy profile $X = (x_1, x_2, \dots, x_n)$ is in equilibrium only if no lone facility have a neighbor facility of the same player.

Proof. A violation of the above condition can appear in two forms:

substructure 1: Two lone facilities of player i are neighbors - she can increase the distance between them (as long as they do not get stuck with other facilities of her opponents) and improve her payoff, hence X is not an equilibrium.

substructure 2: Lone facility a_1 of player i has a paired neighbor a_2 which also belongs to player i - denote a_2 's pair a_3 , the revenue of a_1 as p and the revenue of a_2, a_3 as q . In case a_2 is the right neighbor of a_1 (completely symmetrical if it is the left neighbor), we conclude from Lemma 5 that a_1 must also have at least one left neighbor (Since lone facilities cannot be peripheral). Now:

- (a) If the left neighbor(s) are not hers, she can move a_1 to $a'_1 = a_1 - \epsilon$, so her payoff increases by $\frac{\epsilon}{2}$.
- (b) Otherwise, player i controls one of the two left (paired) neighbors, denoted a_4 . Consider the following change: player n makes a_1 peripheral, so now her payoff from it is $\epsilon > 0$. Next, she move a_2 to $a'_2 = a_2 - \frac{\epsilon}{2}$ and a_4 to $a'_4 = a_4 + \frac{\epsilon}{2}$. The change in her payoff is:

$$\epsilon + \left(q + q + p - \frac{\epsilon}{2} \right) > p + 2q$$

Thus it is a beneficial deviation for her. □

Next, we show that the conditions of the two previous lemmas, when added together, are the sufficient conditions for a profile to be in equilibrium.

Theorem 4. *In a multi-unit game G , necessary and sufficient conditions for a pure strategy profile X to be in equilibrium are:*

1. $\psi(X)$ is in equilibrium in the flattened game G_f .
2. No lone facility has a neighbor facility of the same player under X .

Proof. One direction follows immediately from Lemmas 5,6. On the other direction, assume $X = (X_{-i}, X_i)$ is not in equilibrium, and player i has a profitable deviation. Over all possible profitable deviations, mark X'_i as a deviation with minimum relocations of i 's facilities (discrete number hence well defined), and α as the number of facilities relocated under X'_i . Denote $A = \{a_1, \dots, a_z\}$ as all locations of facilities controlled by i 's rivals under X_{-i} . Observe that since we are given that any lone facility of i does not have neighbors which belongs to i , it follows that under X any facility of i have left and right neighbors from $A \cup \{0, 1\}$ (we treat $\{0, 1\}$ as facilities for simplicity).

Fix a relocated facility of player i . Denote a, a_l, a_r as its location, its left neighbor's location and its right neighbor's location under X . Also, denote a' as its location under X'_i , and $a'_l, a'_r \in A$ as the closest facilities to a' from left and right, out of all other players' facilities. Observe that:

- If no facility of i was relocated into (a_l, a_r) or removed from (a'_l, a'_r) , we get a contradiction of $\psi(X)$ being equilibrium in G_f .
- Since (a'_l, a'_r) contained at most one facility of i (otherwise two lone facilities of i are neighbors), if it was removed we get contradiction to α being minimal.
- If only one facility was relocated into (a_l, a_r) , we get contradiction to α being minimal.

Hence, for each facility removed from an interval, two or more facilities were inserted to the same interval. Since the number of relocations equals the number of removals, the only possible way for that to happen is that for two relocated facilities j', j'' originally located in a, b , the intervals $(a_l, a_r), (b_l, b_r)$ intersect, and a, b are neighbors, hence $a, b \in A$. It also implies that the facilities inserted into (a, b) receives equal or less payoff than what j', j'' received under X , hence contradiction to α being the minimum relocations. □

In Fig. 1 we show two equilibrium profiles for the described game. Note that infinite equilibria can be achieved, not only by swapping the facilities' location, but also by decreasing the distance between the middle facilities, and changing the locations of the peripherals accordingly.

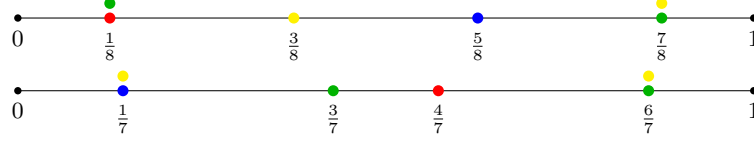


Fig. 1: Two pure equilibria in multi-unit game with $n = 4$ and $(k_b, k_r, k_g, k_y) = (1, 1, 2, 2)$.

4.3 Pure Strategy Equilibrium Existence

Until now, we focused on determining if a profile is an equilibrium for a specific game, and avoided the question of whether this game even possess a pure equilibrium. In this sub-section, we aim to fully classify the set of games which possess pure equilibrium. The notion of dominant player (formally defined below) closely relates to a duel with a stronger player, hence it is only natural to look at it as a dominant-vs-all game. The following notion is crucial for the upcoming theorem:

Definition 2 (Dominant Player). *In a multi-unit game, player n is called a dominant player if she controls more than half of all facilities, i.e. -*

$$\sum_{i \neq n} k_i < k_n$$

Before we deal with any general K , note that possible multi-unit games with $K \leq 3$ are:

1. The original Hotelling setting [7] - 1 vs 1 game, which has a pure equilibrium when both player choose $\frac{1}{2}$.
2. The game of $l = 1$ vs $k = 2$ - which does not have a pure equilibrium (but mixed equilibrium does exist) as seen earlier.
3. Single-unit game with three players - this is known to have no pure equilibrium (see [4]), but mixed equilibrium does exist (as studied in [14]).

Leaving these games aside, it turns out that the sole property one has to know about a multi-unit game in order to determine whether it possesses a pure equilibrium or not, is the existence of a dominant player.

Theorem 5. *In a multi-unit game G with sum of facilities K (where $K \geq 4$), there exists a pure Nash equilibrium iff it has no dominant player.*

Proof. First, consider G with dominant player (w.l.o.g. player n is the dominant) and for simplicity denote $l = \sum_{i \neq n} k_i$. If $k_n \geq 2l$, it is obvious that pure equilibrium cannot exist, since the dominant player can "surround" all of her opponents' facilities, with two of hers, and impose a payoff of $1 - \epsilon$ for any arbitrary $\epsilon > 0$, thus she will never be satisfied. For the rest of the analysis we assume $k_n < 2l$, hence none of the dominant player's facilities are redundant, i.e. every facility of hers serves positive proportion of customers.

Assume by contradiction that there is a pure equilibrium profile X . Since $k_n > l$, there are facilities of player n which are not paired. According to Theorem 4, a lone facility of player n cannot have a neighbor which also belongs to player n . Denote α as the number of player n 's lone facilities. Since each of her lone facility has at least 2 neighbor facilities of players $1, 2, \dots, n-1$, there are at least $\alpha + 1$ such neighbors. In addition, player n has $k_n - \alpha$ paired facilities, none of which are neighbors of her lone facilities, so there are $k_n - \alpha$ disjoint facilities of players other than n that they pair up with. Overall, we get that the number of facilities is at least:

$$\alpha + (\alpha + 1) + (k_n - \alpha) + (k_n - \alpha) = 2k_n + 1 > k + l = K$$

Which yields the desired contradiction.

On the other direction, assume G has no dominant player. We shall construct a profile satisfying the conditions of Theorem 4. If K is even, we can pair all the facilities in $\frac{2i-1}{2k}$ for $i \in \{1, \dots, k\}$, such that each pair is assembled from facilities of different players (this is possible due to the lack of dominant player). The flattened profile is in equilibrium in G_f , and there are no lone facilities, hence the constructed profile meets the sufficient conditions. If K is odd, denote $\alpha = \frac{k+1}{2}$. Again, we arrange all facilities in $\frac{2i-1}{2\alpha}$, and demand that the lone facility is of player 1 (the weakest player) and not peripheral. If player 1 has only one facility, the two conditions of Theorem 4 are easily met. In case she has more than one facility (hence $K \geq 7$), we must show that her remaining facilities can be located such that they will not be neighbors of her lone one. This is possible since:

$$\underbrace{\left(\alpha - 3\right)}_{\text{remaining locations}} - \underbrace{\left\lfloor \frac{K}{3} \right\rfloor - 1}_{\substack{\text{facilities of} \\ \text{player 1 excluding} \\ \text{the lone one}}} = \frac{K}{2} - \left\lfloor \frac{K}{3} \right\rfloor - 1.5 \stackrel{K \geq 7, \text{odd}}{\geq} 0$$

Therefore the above profile also satisfies the sufficient conditions. \square

This concludes the characterization of games with pure equilibrium. Interestingly, we now show conditions under which monopoly-optimal semi-mixed equilibria can be found in the presence of dominant player, extending the result obtained from the l vs k setting. The idea is to choose strategies of players $1, 2, \dots, n-1$ that together mimic the strategy of the weak player in the duel, as the described next. Needless to say, they should also be best response to one another.

Lemma 7. *In a multi-unit game with a dominant player, a semi-mixed equilibrium can be obtained if for every $j \in \{1, 2, \dots, n\}$ it holds that:*

$$b_j = k_j \cdot \frac{k_n}{\sum_{i=1}^{n-1} k_i} \in \mathbb{N}$$

Proof. Denote $l = \sum_{i=1}^{n-1} k_i$. We shall describe the set of mixed strategies $X = (x_1, x_2, \dots, x_n)$, and show that they are in equilibrium. First, the dominant player plays exactly as in l vs k_n :

$$x_n = \left(\frac{1}{2k_n}, \frac{3}{2k_n}, \dots, \frac{2k_n - 1}{2k_n} \right)$$

Next, player j is allocated exactly b_j points occupied with the dominant player's facilities, and her strategy x_j is to choose uniformly a tuple of size k_j out of them (Similar to Protocol 1). Since $\sum_{i=1}^{n-1} b_i = k_n$, it is easy to come up with allocation under which all points are allocated, and no point is allocated to more than one player. The dominant player in fact "sees" a strategy of one player with l facilities, so her strategy is a best response to X_{-n} according to Theorem 2. Furthermore, any facility of a weak player cannot get more than $\frac{1}{2k_n}$, which is exactly what it gets under X , so player j ($j < n$) also plays a best response to X_{-j} , hence X forms an equilibrium. \square

Remark: One might think that a profile X where (the dominant) player n plays the single player's optimal locations, and players $1, 2, \dots, n-1$ strategies' conform with the desired μ (as in Theorem 2) is in fact in equilibrium. This is not always true, since if $b_j \notin \mathbb{N}$, a location α , which populates a facility of player n , will be selected by two players j, i ($i \neq j$) with positive probability. Therefore, player j 's payoff is strictly less than $\frac{k_j}{2k_n}$, and since she can deviate to more profitable strategy (e.g. selecting centers of intervals), X cannot be an equilibrium.

5 Discussion and Future Work

We have presented a natural extension of the pure Hotelling location game, which has many motivating scenarios in the field of Machine Learning, Advertising, etc. For the case of two players, we showed the existence and quasi-uniqueness of equilibrium, where the strong player simply ignores the existence of the weaker player, while the weak player’s strategy is to mix between subsets of the stronger player locations.

Afterwards, we considered a game of multiple players, and extended the necessary and sufficient conditions required for the existence of pure equilibrium in Hotelling game to our setting. We showed that even when a game contains a dominant player, relatively simple, monopoly-optimal semi-mixed equilibrium may exist. The existence of an equilibrium in *any* such game is left as an open question.

One more direction to investigate is what happens when the customers distribution is not uniform, as studied in [10] for the single-unit Hotelling game.

Consider for example the 2 vs 1 duel: Assume that the customers’ preferences are distributed along $(-\infty, \infty)$ according to some (known) distribution function $f_c(t)$. We will denote the cumulative distribution function as $F_c(t)$.⁷ Let q_a^c be the a -th quantile of $F_c(t)$, namely

$$q_a^c = \inf\{t \in (-\infty, \infty) : a \leq F_c(t)\}$$

A simple generalization of the equilibrium protocol introduced earlier is as follows:

$$x_1^c = \begin{cases} q_{1/4}^c & w.p. \frac{1}{2} \\ q_{3/4}^c & w.p. \frac{1}{2} \end{cases}, \quad x_2^c = (q_{1/4}^c, q_{3/4}^c)$$

If the customers’ distribution is Bi-Modal, it can be easily shown that the profile above is in fact in equilibrium. However, in case $f_c(t)$ is the Symmetric Beta distribution, the suggested profile is not in equilibrium, since player 1 would prefer choosing $q_{1/2}^c$ rather than mixing between $q_{1/4}^c$ and $q_{3/4}^c$. In Sect. B we provide more comprehensive analysis of the 2 vs 1 game with non-uniform distributions; We leave the question of equilibrium existence and structure in general multi-unit facility location games under non-uniform distribution open as well.

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⁷ We restrict the scope of this section to continuous functions.

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A Omitted Proofs:

For Sect. 2:

Proof (Of Proposition 1). Let $x_2 = (a_1, a_2 \dots a_k)$ be the (sorted) locations player 2 chose. We also note $a_0 = 0, a_{k+1} = 1$. For $1 \leq i \leq k+1$ we define

$$u_i = a_i - a_{i-1} \quad (4)$$

Instead of analyzing X , we analyze the corresponding $U = (u_1, u_2 \dots u_{k+1})$. Note that $\sum_{j=1}^{k+1} u_j = 1$ must hold, so we use Lagrange multipliers for finding the local minima of the distance function $D(U)$ subject to these equality constraints. For $i \in \{2, 3, \dots k-1\}$, the expected distance of a customer to her nearest facility:

$$\int_{a_{i-1}}^{a_i} \min\{t - a_{i-1}, a_i - t\} \cdot 1 dt = \int_0^{\frac{u_i}{2}} 2t dt = [t^2]_0^{\frac{u_i}{2}} = \frac{u_i^2}{4}$$

and for u_1 (same result for u_{k+1}):

$$\int_0^{u_1} t dt = \left[\frac{t^2}{2}\right]_0^{u_1} = \frac{u_1^2}{2}$$

In total, The expected distance from a random customer to the closest facility is:

$$D(U, \lambda) = \frac{u_1^2}{2} + \sum_{i=2}^k \frac{u_i^2}{4} + \frac{u_{k+1}^2}{2} - \lambda \left(\sum_{j=1}^{k+1} u_j - 1 \right)$$

Taking the partial derivative we get -

$$\begin{cases} \frac{\partial D(U, \lambda)}{\partial u_i} = 0 \Rightarrow u_i = 2\lambda, \\ \frac{\partial D(U, \lambda)}{\partial u_{k+1}} = 0 \Rightarrow u_{k+1} = \lambda \\ \frac{\partial D(U, \lambda)}{\partial u_1} = 0 \Rightarrow u_1 = \lambda \\ \frac{\partial D(U, \lambda)}{\partial \lambda} = 0 \Rightarrow \sum_{j=1}^{k+1} u_j = 1 \end{cases}$$

When we solve these equations we get that the optimal intervals are:

$$u_i = \begin{cases} \frac{1}{2k} & i = 1, k+1 \\ \frac{1}{k} & i \in \{2, 3, \dots k\} \end{cases}$$

Changing the form of equation 4, we get the recurrence relation $a_i = u_i + a_{i-1}$ where u_i is known and $a_0 = 0$. The solution vector is exactly $(\frac{1}{2k}, \dots, \frac{1}{2k}, \dots, \frac{2k-1}{2k})$ \square

Proof (Of Lemma 1). For any deterministic strategy x_1 , a facility $a_i \in x_1$ is either:

1. Located on a facility of player 2. In this case, a_i 's contribution to the payoff of player 1 is $\frac{1}{2k}$ or less.
2. Located between two facilities of player 2. In this case, a_i 's contribute exactly $\frac{1}{2k}$ to player 1's payoff.
3. Located between one facility of player 1 and one facility of player 2. In this case, it contributes less than $\frac{1}{2k}$.
4. Located between two facilities of player 1. In this case, a_i has no contribution to player 1's payoff (namely, player 1 will get the same payoff with this facility).
5. Located between 0 or 1, and facility of player 2. In this case, it contributes less than $\frac{1}{2k}$.

In the first two cases, the contribution of a single facility to player 1's payoff is at most $\frac{1}{2k}$, whereas in the other three cases it is less. Overall, player 1 cannot get more than $\frac{l}{2k}$. \square

For Sect. 3:

Proof (Of Lemma 4).

1. In case player 1 gets less - she can deviate to Protocol 1 and get $\frac{l}{2k}$, and in case she gets more - player 2 can deviate to Protocol 1 and make her get $\frac{l}{2k}$.
2. Any facility which is located in $[0, \frac{1}{2k})$ will give payoff of less than $\frac{1}{2k}$ to player 1, hence contradiction to 1. Similarly for $(\frac{2k-1}{2k}, 1]$.
3. If $U^1(x_1, x_2) > U^1(x_1, x_2^{p_{1k}}) \Rightarrow U^2(x_1, x_2) < U^2(x_1, x_2^{p_{1k}})$, in contradiction to x_1, x_2 being an equilibrium. On the other hand, we saw that $\forall x'_1 : U^1(x'_1, x_2^{p_{1k}}) \leq \frac{l}{2k}$, hence $\frac{l}{2k} = U^1(x_1, x_2) < U^1(x_1, x_2^{p_{1k}}) \leq \frac{l}{2k}$ is also a contradiction.

\square

First, we aim to describe what μ is. A mixed strategy x_1 of player 1 can be thought of as a probability measure P_{x_1} over $([0, 1]^n, \mathcal{B}([0, 1]^n))$. Over the same space, we define a measure μ as follows

$$\mu([a, b]) = \sum_{i=1}^l P_{x_1}(\text{Proj}_i(x_1) \in [a, b])$$

Where Proj_i is the projection of x_1 on the i -th axis. In another words:

$$\mu([a, b]) - \text{the expected number of player 1's facilities in } [a, b]$$

The terms $\mu((a, b))$, $\mu((a, b])$ and $\mu([a, b))$ are also valid, and for simplicity we use $\mu(z)$ to denote the probability of seeing a facility of player 1 under x_1 in z . Obviously, P_{x_1} is induced by x_1 , and therefore μ is also induced by x_1 , so we should have denoted it as μ_{x_1} . However, since the context x_1 is clear whenever we use it, we simply denote this induced measure as μ .

Lemma 8. *For any strategy x_1 of player 1 the following holds:*

1. μ induced by x_1 is a measure.
2. For any $x_1, a \in [0, 1)$ and $\epsilon > 0$ there exists $\delta > 0$ such that:

$$\mu((a, a + \delta)) < \epsilon$$

3. For any $0 \leq a < b \leq 1$ there exists $\delta > 0$ such that $a + \delta < b$ and $\mu(a + \delta) = 0$

Proof.

1. In order to show that μ is a measure we need to show:

(a) Non-negativity: $\forall A \in \mathcal{B}([0, 1])$

$$\mu(A) = \sum_{i=1}^l P_{x_1}(Proj_i(x_1) \in A) \stackrel{P \text{ is a measure}}{\geq} 0$$

(b) Null empty set:

$$\mu(\phi) = \sum_{i=1}^l P_{x_1}(Proj_i(x_1) \in \phi) = \sum_{i=1}^l P_{x_1}(\phi) \stackrel{P \text{ is a measure}}{=} 0$$

(c) Countable additivity: for disjoint set $A_1, A_2, \dots \in \mathcal{B}([0, 1])$:

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{i=1}^l P(proj_i(x_1) \in \bigcup_{i=1}^{\infty} A_i) \stackrel{P \text{ is a measure}}{=} \sum_{i=1}^l \sum_{j=1}^{\infty} P(proj_i(x_1) \in A_j) \\ &= \sum_{j=1}^{\infty} \mu(A_j) \end{aligned}$$

2. We define a decreasing sequence of sets $B_i \in \mathcal{B}([0, 1])$:

$$B_i = \left(a, a + \frac{1}{n_0 + i}\right)$$

Where n_0 is the minimal integer satisfying $a + \frac{1}{n_0} \leq 1$. Since $B_i \downarrow \phi$ and μ is a measure, we know that $\mu(B_i) \downarrow \mu(\phi) = 0$ also holds, and therefore there exists j such that:

$$\mu(B_j) = \mu\left(a, a + \frac{1}{n_0 + j}\right) < \epsilon$$

Hence we denote $\delta = \frac{1}{n_0 + j}$ and the claim holds.

3. Assume by contradiction that for any δ such that $0 < \delta < b - a$ it holds that $\mu(a + \delta) > 0$. We look at the closed interval:

$$\left[\frac{b-a}{4}, \frac{b-a}{2}\right] \subset (0, b-a)$$

Since it is closed and μ is bounded, μ gets minimum inside it, hence we denote:

$$c = \min_{y \in \left[\frac{b-a}{4}, \frac{b-a}{2}\right]} \mu(a + y)$$

But now:

$$\mu\left(\left[\frac{b-a}{4}, \frac{b-a}{2}\right]\right) \geq \sum_{n=1}^{\infty} \mu\left(\frac{b-a}{4} \cdot \left(1 + \frac{1}{n}\right)\right) \geq \sum_{n=1}^{\infty} c = \infty$$

In contradiction to $\mu\left(\left[\frac{b-a}{4}, \frac{b-a}{2}\right]\right) \leq \mu([0, 1]) = l$

□

B Extensions to Non-Uniform Distributions

In this section we relax the assumption of uniform distribution over the customers. We will assume customers' preferences are distributed along $(-\infty, \infty)$ according to some (known) distribution function $f_c(t)$. We will denote the cumulative distribution function as $F_c(t)$. We restrict the scope of this section to continuous functions. Some discrete settings will be briefly mentioned in the discussion section. Our focus is on the extension of the foundational 1 vs 2 setting.

Definition 3. Assume there are 2 facilities a, b located on the segment $[0, 1]$. We say that a point c is an indifference point if a customer located at c is indifferent between going to a or going to b . This is equivalent to:

$$|c - a| = |c - b|$$

Lemma 9. If one player with two facilities aims to maximize the social welfare, she should locate her facilities on a, b ($a \leq b$) such that -

$$\begin{cases} P(t \leq a) = P(a \leq t \leq \frac{a+b}{2}) \\ P(\frac{a+b}{2} \leq t \leq b) = P(t \geq b) \end{cases}$$

We omit the proof of this lemma. A simple explanation to this result is that the optimal locations for a, b turns $c = \frac{a+b}{2}$ into an indifference point, and a, b are the medians of $[0, c]$, $[c, 1]$ respectively.

B.1 Equilibrium in the Non-Uniform Case: LM Distributions.

Denote q_a^c the a -th quantile of $F_c(t)$, namely $q_a^c = \inf\{t \in [-\infty, \infty] : a \leq F_c(t)\}$. A simple generalization of the equilibrium protocol introduced earlier is as follows:

Protocol 2

1. Player 1 chooses her facility's location using

$$x_1^{p3} = \begin{cases} q_{1/4}^c & w.p. \frac{1}{2} \\ q_{3/4}^c & w.p. \frac{1}{2} \end{cases}$$

2. Player 2 locates her facilities deterministically $x_2^{p3} = (q_{1/4}^c, q_{3/4}^c)$.

However, Protocol 2 is not guaranteed to hold as an equilibrium for all possible customer preferences distributions. For example, as will be implied by our general result in the following sub-section it does not hold as equilibrium for $Beta(\alpha, \alpha)$. Nevertheless, we can characterize a large family of natural distributions for which Protocol 2 is in fact equilibrium.

Definition 4. Light-Middle distribution (or simply LM)

A distribution $f_c(t)$ is LM if it satisfies:

1. $\frac{q_{1/4}^c + q_{3/4}^c}{2} = q_{1/2}^c$
2. $\forall x \in (q_{1/4}^c, q_{3/4}^c) : \int_{\frac{q_{1/4}^c + x}{2}}^{\frac{q_{3/4}^c + x}{2}} f_c(t) dt \leq \frac{1}{4}$

An example of LM distribution is provided in figure 2 (a). On first sight, the second condition may appear to be quite technical. The goal behind the next lemma is to point to a simple sufficient condition on when a given customers' preferences distribution is LM, pointing to its generality.

Lemma 10. *If $f_c(t)$ satisfies:*

1. *First condition of LM distribution*
2. *$f_c(t)$ is monotonically non increasing in $[q_{1/4}^c, q_{1/2}^c]$*
3. *$f_c(t)$ is monotonically non decreasing in $[q_{1/2}^c, q_{3/4}^c]$*

Then $f_c(t)$ is a LM distribution

Proof. Denote:

$$y \in \arg \max_{x \in [q_{1/4}^c, q_{3/4}^c]} \left[\int_{\frac{q_{1/4}^c+x}{2}}^{\frac{q_{3/4}^c+x}{2}} f_c(t) dt \right]$$

The integral inside the arg max is continuous in the closed and bounded interval $[q_{1/4}^c, q_{3/4}^c]$, hence by the Extreme Value Theorem it must attain a maximum at least once, so the arg max is well defined.

Assume by contradiction that $\int_{\frac{q_{1/4}^c+y}{2}}^{\frac{q_{3/4}^c+y}{2}} f_c(t) dt > \frac{1}{4}$. From the first property of LM distribution and definition of quantiles we know that $y \notin \{q_{1/4}^c, q_{3/4}^c\}$. There are 3 possible cases:

1. $f_c(\frac{q_{3/4}^c+y}{2}) > f_c(\frac{q_{1/4}^c+y}{2})$: since $y < q_{3/4}^c$, there exists $\epsilon > 0$ such that $y + 2\epsilon < q_{3/4}^c$. Therefore:

$$\begin{aligned} \int_{\frac{q_{1/4}^c+y+2\epsilon}{2}}^{\frac{q_{3/4}^c+y+2\epsilon}{2}} f_c(t) dt &= \int_{\frac{q_{1/4}^c+y}{2}}^{\frac{q_{3/4}^c+y}{2}} f_c(t) dt + \underbrace{\int_{\frac{q_{3/4}^c+y}{2}}^{\frac{q_{3/4}^c+y+2\epsilon}{2}} f_c(t) dt}_{\text{none-decreasing}} - \underbrace{\int_{\frac{q_{1/4}^c+y}{2}}^{\frac{q_{1/4}^c+y+2\epsilon}{2}} f_c(t) dt}_{\text{none-increasing}} \\ &\geq \int_{\frac{q_{1/4}^c+y}{2}}^{\frac{q_{3/4}^c+y}{2}} f_c(t) dt + \underbrace{\epsilon \left[f_c(\frac{q_{3/4}^c+y}{2}) - f_c(\frac{q_{1/4}^c+y}{2}) \right]}_{>0} > \int_{\frac{q_{1/4}^c+y}{2}}^{\frac{q_{3/4}^c+y}{2}} f_c(t) dt \end{aligned}$$

In contradiction to $y \in \arg \max_{x \in [q_{1/4}^c, q_{3/4}^c]} \left[\int_{\frac{q_{1/4}^c+x}{2}}^{\frac{q_{3/4}^c+x}{2}} f_c(t) dt \right]$.

2. $f_c(\frac{q_{1/4}^c+y}{2}) > f_c(\frac{q_{3/4}^c+y}{2})$: since $q_{1/4}^c < y$, there exists $\epsilon > 0$ such that $q_{1/4}^c < y - 2\epsilon$. From here, similar to the previous case.
3. $f_c(\frac{q_{1/4}^c+y}{2}) = f_c(\frac{q_{3/4}^c+y}{2})$: From the monotonicity we get that $\forall t \in [\frac{q_{1/4}^c+y}{2}, \frac{q_{3/4}^c+y}{2}]$ it holds that:

$$\begin{aligned} f_c(\frac{q_{1/4}^c+y}{2}) &\geq f_c(t) \xrightarrow{\text{integrate}} \int_{\frac{q_{1/4}^c+y}{2}}^{\frac{q_{3/4}^c+y}{2}} f_c(\frac{q_{1/4}^c+y}{2}) dt \geq \int_{\frac{q_{1/4}^c+y}{2}}^{\frac{q_{3/4}^c+y}{2}} f_c(t) dt \\ &\Rightarrow (\frac{q_{3/4}^c - q_{1/4}^c}{2}) f_c(\frac{q_{1/4}^c+y}{2}) \geq \int_{\frac{q_{1/4}^c+y}{2}}^{\frac{q_{3/4}^c+y}{2}} f_c(t) dt > \frac{1}{4} \\ &\Rightarrow (\frac{q_{3/4}^c - q_{1/4}^c}{2}) f_c(\frac{q_{1/4}^c+y}{2}) > \frac{1}{4} \end{aligned}$$

Therefore -

$$\begin{aligned}
\int_{q_{1/4}^c}^{q_{3/4}^c} f_c(t) dt &= \int_{q_{1/4}^c}^{\frac{q_{1/4}^c + y}{2}} f_c(t) dt + \int_{\frac{q_{1/4}^c + y}{2}}^{\frac{q_{3/4}^c + y}{2}} f_c(t) dt + \int_{\frac{q_{3/4}^c + y}{2}}^{q_{3/4}^c} f_c(t) dt \\
&> \underbrace{\int_{q_{1/4}^c}^{\frac{q_{1/4}^c + y}{2}} f_c(t) dt}_{\text{none-increasing}} + \underbrace{\int_{\frac{q_{3/4}^c + y}{2}}^{q_{3/4}^c} f_c(t) dt}_{\text{none-decreasing}} + \frac{1}{4} \\
&\geq \left(\frac{q_{1/4}^c + y}{2} - q_{1/4}^c \right) f_c\left(\frac{q_{1/4}^c + y}{2}\right) + \left(q_{3/4}^c - \frac{q_{3/4}^c + y}{2} \right) f_c\left(\frac{q_{3/4}^c + y}{2}\right) + \frac{1}{4} \\
&= \left(\frac{q_{3/4}^c - q_{1/4}^c}{2} \right) f_c\left(\frac{q_{1/4}^c + y}{2}\right) + \frac{1}{4} > \frac{1}{2}
\end{aligned}$$

But $(q_{1/4}^c, q_{3/4}^c)$ are the quantiles, therefore we got a contradiction to $\int_{q_{1/4}^c}^{q_{3/4}^c} f_c(t) dt = \frac{1}{2}$. \square

The meaning of LM is that customers generally have a strong preferences towards one of the "sides" of the segment. More generally, there is no dense subset of the customers hidden between the 1/4 quantile and the 3/4 quantile. This general set of distributions do allow for the following:

Theorem 6 (1 vs 2 equilibrium with LM distribution). *Protocol 2 is an equilibrium for 1 vs 2 facility location with any LM customers' distribution.*

The proof of this theorem relies on the next two lemmas:

Lemma 11. *If the customers distribution is LM, and player 2 plays according to Protocol 2, any deterministic strategy played by player 1 can lead to maximum expected reward of $\frac{1}{4}$.*

Proof. Denote x_1 as the strategy of player 1. For all possible values of x_1 , we look at $U^1(x_1, x_2^{p3})$:

$$\begin{aligned}
(1) \quad x_1 < q_{1/4}^c &: \int_{-\infty}^{\frac{x_1 + q_{1/4}^c}{2}} f_c(t) dt < \int_{-\infty}^{q_{1/4}^c} f_c(t) dt = \frac{1}{4} \\
(2) \quad x_1 = q_{1/4}^c &: \frac{1}{2} \int_{-\infty}^{\frac{q_{1/4}^c + q_{3/4}^c}{2}} f_c(t) dt \stackrel{\text{property 1}}{=} \frac{1}{2} \int_{-\infty}^{q_{1/2}^c} f_c(t) dt = \frac{1}{4} \\
(3) \quad q_{1/4}^c < x_1 \leq q_{1/2}^c &: \underbrace{\int_{\frac{q_{1/4}^c + x_1}{2}}^{\frac{q_{3/4}^c + x_1}{2}} f_c(t) dt}_{\text{property 2}} \leq \frac{1}{4}
\end{aligned}$$

And similarly for the cases we omitted. \square

Lemma 12. *If the customers preferences' distribution is LM, and player 1 plays according to Protocol 2, any deterministic strategy played by player 2 can lead to maximum expected reward of $\frac{3}{4}$.*

Proof. We know that player 1 plays according to x_1^{p3} . We denote $x_2 = (a, b)$ as the deterministic strategy where player 2 locates one facility on a and one facility on b (w.l.o.g. $a \leq b$). We create a partition of $[-\infty, \infty]$, and consider all possible cases. Denote

$$I_1 = (-\infty, q_{1/4}^c), I_2 = \{q_{1/4}^c\}, I_3 = (q_{1/4}^c, q_{3/4}^c), I_4 = \{q_{3/4}^c\}, I_5 = (q_{3/4}^c, \infty)$$

and let $L(l) = i$ such that $l \in I_i$. The possible cases for $(L(a), L(b))$ are:

$$\underbrace{(1, 1), (5, 5)}_1, \underbrace{(1, 2), (4, 5)}_2, \underbrace{(3, 3)}_3, \underbrace{(2, 3), (3, 4)}_4, \underbrace{(1, 3), (3, 5)}_5, \underbrace{(1, 4), (2, 5)}_6, \underbrace{(1, 5)}_7, \underbrace{(2, 2), (4, 4)}_8, \underbrace{(2, 4)}_9$$

Two cases marked with the same digit are completely symmetrical, so we show only one of them (the left one). For each case, we look at $U^1(x_1^{p_3}, x_2)$:

$$(1) > \frac{1}{2} \int_{q_{1/4}^c}^{\infty} f_c(t) dt + \frac{1}{2} \int_{q_{1/2}^c}^{\infty} f_c(t) dt = \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{8}$$

$$(2) = \frac{1}{2} \cdot \frac{1}{2} \int_{q_{1/4}^c}^{\infty} f_c(t) dt + \frac{1}{2} \int_{q_{1/2}^c}^{\infty} f_c(t) dt = \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{7}{16}$$

$$(3) > \frac{1}{2} \int_{-\infty}^{q_{1/4}^c} f_c(t) dt + \frac{1}{2} \int_{q_{3/4}^c}^{\infty} f_c(t) dt = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4}$$

$$(4) = \frac{1}{2} \cdot \frac{1}{2} \int_{-\infty}^{\frac{b+q_{1/4}^c}{2}} f_c(t) dt + \frac{1}{2} \int_{\frac{b+q_{3/4}^c}{2}}^{\infty} f_c(t) dt$$

$$= \frac{1}{2} \cdot \left(\frac{1}{2} \int_{-\infty}^{\frac{b+q_{1/4}^c}{2}} f_c(t) dt + \int_{\frac{b+q_{1/4}^c}{2}}^{\infty} f_c(t) dt - \int_{\frac{b+q_{1/4}^c}{2}}^{\frac{b+q_{1/4}^c}{2}} f_c(t) dt \right)$$

$$\stackrel{\text{property 2}}{\geq} \frac{1}{2} \cdot \left(\int_{-\infty}^{\infty} f_c(t) dt - \frac{1}{2} \int_{-\infty}^{\frac{b+q_{1/4}^c}{2}} f_c(t) dt - \frac{1}{4} \right)$$

$$= \frac{1}{2} \cdot \left(\frac{3}{4} - \frac{1}{2} \int_{-\infty}^{\frac{b+q_{1/4}^c}{2}} f_c(t) dt \right) = \frac{3}{8} - \frac{1}{4} \int_{-\infty}^{\frac{b+q_{1/4}^c}{2}} f_c(t) dt$$

$$\text{Recall that } b < q_{3/4}^c \Rightarrow \frac{b+q_{1/4}^c}{2} < \frac{q_{3/4}^c + q_{1/4}^c}{2} = q_{1/2}^c :$$

$$= \frac{3}{8} - \frac{1}{4} \int_{-\infty}^{\frac{b+q_{1/4}^c}{2}} f_c(t) dt \geq \frac{3}{8} - \frac{1}{4} \int_{-\infty}^{q_{1/2}^c} f_c(t) dt = \frac{1}{4}$$

$$(5) = \frac{1}{2} \int_{\frac{a+q_{1/4}^c}{2}}^{\frac{b+q_{1/4}^c}{2}} f_c(t) dt + \frac{1}{2} \int_{\frac{b+q_{3/4}^c}{2}}^{\infty} f_c(t) dt = \frac{1}{2} \cdot \left(\int_{\frac{a+q_{1/4}^c}{2}}^{\infty} f_c(t) dt - \int_{\frac{b+q_{1/4}^c}{2}}^{\frac{b+q_{3/4}^c}{2}} f_c(t) dt \right)$$

$$\geq \frac{1}{2} \cdot \left(\frac{3}{4} - \int_{\frac{b+q_{1/4}^c}{2}}^{\frac{b+q_{3/4}^c}{2}} f_c(t) dt \right) \stackrel{\text{property 2}}{\geq} \frac{1}{2} \cdot \left(\frac{3}{4} - \frac{1}{4} \right) = \frac{1}{4}$$

$$(6) = \frac{1}{2} \int_{\frac{a+q_{1/4}^c}{2}}^{\frac{q_{1/4}^c + q_{3/4}^c}{2}} f_c(t) dt + \frac{1}{2} \cdot \frac{1}{2} \int_{\frac{a+q_{3/4}^c}{2}}^{\infty} f_c(t) dt \stackrel{\text{property 1}}{=} \frac{1}{2} \int_{\frac{a+q_{1/4}^c}{2}}^{q_{1/2}^c} f_c(t) dt + \frac{1}{2} \cdot \frac{1}{2} \int_{\frac{x+q_{3/4}^c}{2}}^{\infty} f_c(t) dt$$

$$\geq \frac{1}{2} \int_{q_{1/4}^c}^{q_{1/2}^c} f_c(t) dt + \frac{1}{2} \cdot \frac{1}{2} \int_{q_{1/2}^c}^{\infty} f_c(t) dt \geq \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$(7) > \frac{1}{2} \int_{q_{1/4}^c}^{\frac{q_{1/4}^c + b}{2}} f_c(t) dt + \frac{1}{2} \int_{\frac{q_{3/4}^c + a}{2}}^{q_{3/4}^c} f_c(t) dt \geq \frac{1}{2} \int_{q_{1/4}^c}^{q_{1/2}^c} f_c(t) dt + \frac{1}{2} \int_{q_{1/2}^c}^{q_{3/4}^c} f_c(t) dt$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$(8) = \frac{1}{2} \cdot \frac{1}{2} \int_{-\infty}^{\infty} f_c(t) dt + \frac{1}{2} \int_{q_{1/2}^c}^{\infty} f_c(t) dt = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$(9) = \frac{1}{2} \cdot \frac{1}{2} \int_{-\infty}^{q_{1/2}^c} f_c(t) dt + \frac{1}{2} \cdot \frac{1}{2} \int_{q_{1/2}^c}^{\infty} f_c(t) dt = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Therefore it holds that for all possible strategies x_2 $U^1(x_1^{p_3}, x_2) \geq \frac{1}{4}$, and because this is a 1-sum game we know that $U^2(x_1^{p_3}, x_2) \leq \frac{3}{4}$. \square

Proof (Of Theorem 6). Lemmas 11 and 12 imply that any change in player i 's strategy will lead to decrease in her expected reward, and therefore it is an equilibrium by definition.

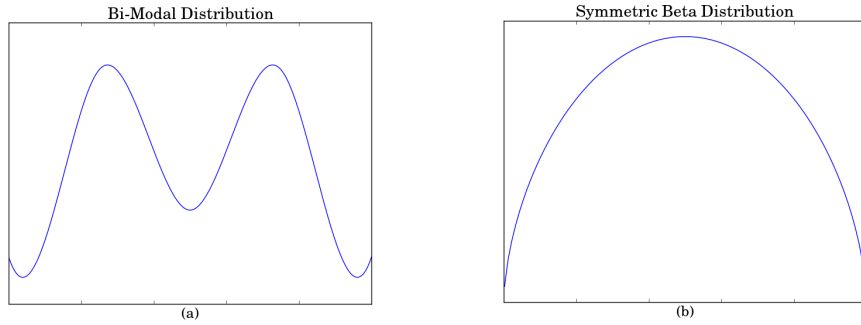


Fig. 2: Examples of a LM (a) and a MHM (b) distributions

B.2 The MHM Distributions.

Next, we shall analyze a family of natural distributions where Protocol 2 does not hold as equilibrium. Our result will show that selecting among the optimal single-player facility locations will not hold as equilibrium.

Definition 5. A distribution is *Monotone-Heavy-Middle* (or simply *MHM*) if it satisfies:

1. $\frac{q_{1/4}^c + q_{3/4}^c}{2} = q_{1/2}^c$
2. $f_c(t)$ is monotonically increasing in $[q_{1/4}^c, q_{1/2}^c]$
3. $f_c(t)$ is monotonically decreasing in $[q_{1/2}^c, q_{3/4}^c]$

A wide range of classical preferences distributions are MHM - Normal, Beta etc (see figure 2 (b) for an illustration). The situations MHM captures are where the majority of people does not hold a strong preference towards the sides, but rather enjoys being the median. In a sense, such distributions are the complement of the LM distributions.

One may wonder if there are any locations, not necessarily the single-player optimal locations as in Protocol 2, where player 2 can locate her facilities and player 1 act in a similar manner (i.e. select between player 2's facilities). The following lemma shows that this situation is impossible:

Lemma 13. *There are no non-strictly-mixed equilibria for any MHM distribution.*

Proof. Since there is neither a pure equilibrium nor an equilibrium where player 1 plays pure strategy and player 2 play mixed strategy, we are left with the case where player 2 plays a deterministic strategy (a, b) , and player 1 selects between these locations with some probabilities p_a, p_b . Assume by contradiction that a profile of such strategies (x_1, x_2) is an equilibrium. Using the fact that $U^1(a, (a, b)) = U^1(b, (a, b))$ we conclude that:

$$\frac{1}{2}F_c\left(\frac{a+b}{2}\right) = \frac{1}{2} \cdot \left(1 - F_c\left(\frac{a+b}{2}\right)\right) \Rightarrow \frac{a+b}{2} = q_{1/2}^c$$

Hence $U^1(x_1, x_2) = \frac{1}{4}$. Furthermore, Since (x_1, x_2) is an equilibrium profile, player 1 does not deviate, hence $\forall \epsilon > 0$ it holds that:

$$F_c(a) - \epsilon < \frac{1}{2}F_c\left(\frac{a+b}{2}\right) \Rightarrow a \leq q_{1/4}^c$$

And using similar argument we conclude that $b \geq q_{3/4}^c$. By definition of MHM $f_c(t)$ is monotonically decreasing in $[q_{1/2}^c, q_{3/4}^c]$, thus:

$$\begin{aligned} \forall t \in [0, \frac{q_{3/4}^c - q_{1/2}^c}{2}] : f_c(q_{1/2}^c + t) &> f_c\left(\frac{q_{1/2}^c + q_{3/4}^c}{2} + t\right) \\ \Rightarrow \int_0^{\frac{q_{3/4}^c - q_{1/2}^c}{2}} f_c(q_{1/2}^c + t) dt &> \int_0^{\frac{q_{3/4}^c - q_{1/2}^c}{2}} f_c\left(\frac{q_{1/2}^c + q_{3/4}^c}{2} + t\right) dt \\ \Rightarrow F_c\left(\frac{q_{1/2}^c + q_{3/4}^c}{2}\right) - F_c(q_{1/2}^c) &> F_c(q_{3/4}^c) - F_c\left(\frac{q_{1/2}^c + q_{3/4}^c}{2}\right) \Rightarrow F_c\left(\frac{q_{1/2}^c + q_{3/4}^c}{2}\right) > \frac{5}{8} \end{aligned}$$

And similarly we show that $F_c\left(\frac{q_{1/2}^c + q_{1/4}^c}{2}\right) < \frac{3}{8}$. Now, if player 1 deviates and plays $x'_1 = q_{1/2}^c$, her payoff will be:

$$F_c\left(\frac{b + q_{1/2}^c}{2}\right) - F_c\left(\frac{a + q_{1/2}^c}{2}\right) \geq F_c\left(\frac{q_{1/2}^c + q_{3/4}^c}{2}\right) - F_c\left(\frac{q_{1/2}^c + q_{1/4}^c}{2}\right) > \frac{1}{4}$$

In contradiction to (x_1, x_2) being an equilibrium profile. □

As an immediate result it follows that Protocol 2 does not hold as equilibrium in MHM distributions. The question of whether for a given MHM distribution there is any equilibrium or not remains open.